A FORMULATION OF THE SIMPLE THEORY OF TYPES

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The purpose of the present paper is to give a formulation of the simple theory of types[1] which incorporates certain features of the calculus of λ-conversion.[2] A complete incorporation of the calculus of λ-conversion into the theory of types is impossible if we require that λx and juxtaposition shall retain their respective meanings as an abstraction operator and as denoting the application of function to argument. But the present partial incorporation has certain advantages from the point of view of type theory and is offered as being of interest on this basis (whatever may be thought of the finally satisfactory character of the theory of types as a foundation for logic and mathematics).

For features of the formulation which are not immediately connected with the incorporation of λ-conversion, we are heavily indebted to Whitehead and Russell,[3] Hilbert and Ackermann,[4] Hilbert and Bernays,[5] and to forerunners of these, as the reader familiar with the works in question will recognize.

1. The hierarchy of types. The class of *type symbols* is described by the rules that ι and o are each type symbols and that if α and β are type symbols then $(\alpha\beta)$ is a type symbol: it is the least class of symbols which contains the symbols ι and o and is closed under the operation of forming the symbol $(\alpha\beta)$ from the symbols α and β.

As exemplified in the statement just made, we shall use the Greek letters α, β, γ to represent variable or undetermined type symbols. We shall abbreviate type symbols by omission of parentheses with the convention that association is to the left—so that, for instance, $o\iota$ will be an abbreviation for $(o\iota)$, $\iota\iota\iota$ for $((\iota\iota)\iota)$, $\iota\iota(\iota\iota)$ for $((\iota\iota)(\iota\iota))$, etc. Moreover, we shall use α' as an abbreviation for $((\alpha\alpha)(\alpha\alpha))$, α'' as an abbreviation for $((\alpha'\alpha')(\alpha'\alpha'))$, etc.

The type symbols enter our formal theory only as subscripts upon variables and constants. In the interpretation of the theory it is intended that the

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[1] See Rudolf Carnap, ***Abriss der Logistik,*** Vienna 1929, §9. (The simple theory of types was suggested as a modification of Russell's ramified theory of types by Leon Chwistek in 1921 and 1922 and by F. P. Ramsey in 1926.)

[2] See, for example, Alonzo Church, ***Mathematical logic*** (mimeographed), Princeton, N. J., 1936, and ***The calculi of lambda-conversion,*** forthcoming monograph.

[3] Bertrand Russell, *Mathematical logic as based on the theory of types*, ***American journal of mathematics,*** vol. 30 (1908), pp. 222-262; Alfred North Whitehead and Bertrand Russell, ***Principia mathematica,*** vol. 1, Cambridge, England, 1910 (second edition 1925), vol. 2, Cambridge, England, 1912 (second edition 1927), and vol. 3, Cambridge, England, 1913 (second edition 1927).

[4] D. Hilbert and W. Ackermann, ***Grundzüge der theoretischen Logik,*** Berlin 1928 (second edition 1938).

[5] D. Hilbert and P. Bernays, ***Grundlagen der Mathematik,*** vol. 1, Berlin 1934, and vol. 2, Berlin 1939.

subscript shall indicate the type of the variable or constant, o being the type of propositions, ι the type of individuals, and $(\alpha\beta)$ the type of functions of one variable for which the range of the independent variable comprises the type β and the range of the dependent variable is contained in the type α. Functions of several variables are explained, after Schönfinkel,[6] as functions of one variable whose values are functions, and propositional functions are regarded simply as functions whose values are propositions. Thus, e.g., $o\iota\iota$ is the type of propositional functions of two individual variables.

We purposely refrain from making more definite the nature of the types o and ι, the formal theory admitting of a variety of interpretations in this regard. Of course the matter of interpretation is in any case irrelevant to the abstract construction of the theory, and indeed other and quite different interpretations are possible (formal consistency assumed).

2. Well-formed formulas. The *primitive symbols* are given in the following infinite list:

$$\lambda,\ (,\ ),\ N_{oo},\ A_{ooo},\ \Pi_{o(o\alpha)},\ \iota_{\alpha(o\alpha)},\ a_\alpha,\ b_\alpha,\ \cdots,\ z_\alpha,\ \bar{a}_\alpha,\ \bar{b}_\alpha,\ \cdots.$$

Of these, the first three are *improper symbols*, and the others are *proper symbols*. Of the proper symbols, N_{oo}, A_{ooo}, $\Pi_{o(o\alpha)}$, and $\iota_{\alpha(o\alpha)}$ are *constants*, and the remainder are *variables*.

(The inclusion of $\Pi_{o(o\alpha)}$ in this list of primitive symbols is meant in this sense, that, if α is any type symbol, $\Pi_{o(o\alpha)}$ is a primitive symbol, a proper symbol, and a constant; similarly in the case of $\iota_{\alpha(o\alpha)}$, a_α, etc.)

Any finite sequence of primitive symbols is a *formula*. Certain formulas are distinguished as being *well-formed* and as having a certain *type*, in accordance with the following rules: (1) a formula consisting of a single proper symbol is well-formed and has the type indicated by the subscript; (2) if $\mathbf{x}_\beta$ is a variable with subscript β and $\mathbf{M}_\alpha$ is a well-formed formula of type α, then $(\lambda \mathbf{x}_\beta \mathbf{M}_\alpha)$ is a well-formed formula having the type $\alpha\beta$; (3) if $\mathbf{F}_{\alpha\beta}$ and $\mathbf{A}_\beta$ are well-formed formulas of types $\alpha\beta$ and β respectively, then $(\mathbf{F}_{\alpha\beta}\mathbf{A}_\beta)$ is a well-formed formula having the type α. The well-formed formulas are the least class of formulas which these rules allow, and the type of a well-formed formula is that determined (uniquely) by these rules. An occurrence of a variable $\mathbf{x}_\beta$ in a well-formed formula is *bound* or *free* according as it is or is not an occurrence in a well-formed part of the formula having the form $(\lambda \mathbf{x}_\beta \mathbf{M}_\alpha)$. The bound variables of a well-formed formula are those which have bound occurrences in the formula, and the free variables are those which have free occurrences.

In making metamathematical (syntactical) statements, we shall use bold capital letters as variables for well-formed formulas, and bold small letters as variables for variables, employing subscripts to denote the type—as in the preceding paragraph. Moreover we shall adopt the customary, self-explanatory, usage, according to which symbols belonging to the formal language serve in

[6] M. Schönfinkel, *Über die Bausteine der mathematischen Logik*, ***Mathematische Annalen***, vol. 92 (1924), pp. 305–316.

the syntax language (English) as names for themselves, and juxtaposition serves to denote juxtaposition.

In writing well-formed formulas we shall often employ various conventions of abbreviation. In particular, we may omit parentheses ( ) when possible without ambiguity, using the convention in restoring omitted parentheses that the formula must be well-formed and that otherwise association is to the left. Thus, for instance, $a_{\iota'} b_{\iota\iota}(c_{\iota\iota} d_{\iota})$ is an abbreviation for $((a_{((\iota\iota)(\iota\iota))} b_{(\iota\iota)})(c_{\iota\iota} d_{\iota}))$, and $\lambda b_{\iota\iota} \lambda c_{\iota\iota}(a_{\iota'} b_{\iota\iota}(c_{\iota\iota} d_{\iota}))$ is an abbreviation for $(\lambda b_{\iota\iota}(\lambda c_{\iota\iota}((a_{((\iota\iota)(\iota\iota))} b_{(\iota\iota)})(c_{\iota\iota} d_{\iota}))))$.

As indicated in the examples just given, type-symbol subscripts may be abbreviated in the way described in §1. When the subscript is o it may be omitted altogether: thus a small italic letter without subscript is to be read as having the subscript o.

We introduce further the following conventions of abbreviation (reading the arrow as "stands for," or "is an abbreviation for"):

$[\sim A_o] \rightarrow N_{oo} A_o.$
$[A_o \vee B_o] \rightarrow A_{ooo} A_o B_o.$
$[A_o B_o] \rightarrow [\sim[[\sim A_o] \vee [\sim B_o]]].$
$[A_o \supset B_o] \rightarrow [[\sim A_o] \vee B_o].$
$[A_o \equiv B_o] \rightarrow [[A_o \supset B_o][B_o \supset A_o]].$
$[(x_\alpha) A_o] \rightarrow \Pi_{o(o\alpha)}(\lambda x_\alpha A_o).$
$[(\exists x_\alpha) A_o] \rightarrow [\sim[(x_\alpha)[\sim A_o]]].$
$[(\imath x_\alpha) A_o] \rightarrow \iota_{\alpha(o\alpha)}(\lambda x_\alpha A_o).$
$Q_{o\alpha\alpha} \rightarrow \lambda x_\alpha \lambda y_\alpha[(f_{o\alpha})[f_{o\alpha} x_\alpha \supset f_{o\alpha} y_\alpha]].$
$[A_\alpha = B_\alpha] \rightarrow Q_{o\alpha\alpha} A_\alpha B_\alpha.$
$[A_\alpha \neq B_\alpha] \rightarrow [\sim[A_\alpha = B_\alpha]].$
$I_{\alpha\alpha} \rightarrow \lambda x_\alpha x_\alpha.$
$K_{\alpha\beta\alpha} \rightarrow \lambda x_\alpha \lambda y_\beta x_\alpha.$
$0_{\alpha'} \rightarrow \lambda f_{\alpha\alpha} \lambda x_\alpha x_\alpha,$
$1_{\alpha'} \rightarrow \lambda f_{\alpha\alpha} \lambda x_\alpha(f_{\alpha\alpha} x_\alpha),$
$2_{\alpha'} \rightarrow \lambda f_{\alpha\alpha} \lambda x_\alpha(f_{\alpha\alpha}(f_{\alpha\alpha} x_\alpha)),$
$3_{\alpha'} \rightarrow \lambda f_{\alpha\alpha} \lambda x_\alpha(f_{\alpha\alpha}(f_{\alpha\alpha}(f_{\alpha\alpha} x_\alpha)))$, etc.
$S_{\alpha'\alpha'} \rightarrow \lambda n_{\alpha'} \lambda f_{\alpha\alpha} \lambda x_\alpha(f_{\alpha\alpha}(n_{\alpha'} f_{\alpha\alpha} x_\alpha)).$
$N_{o\alpha'} \rightarrow \lambda n_{\alpha'}[(f_{o\alpha'})[f_{o\alpha'} 0_{\alpha'} \supset [[(x_{\alpha'})[f_{o\alpha'} x_{\alpha'} \supset f_{o\alpha'}(S_{\alpha'\alpha'} x_{\alpha'})]] \supset f_{o\alpha'} n_{\alpha'}]]].$
$\omega_{\alpha''\alpha'\alpha'} \rightarrow \lambda y_{\alpha'} \lambda z_{\alpha'} \lambda f_{\alpha'\alpha'} \lambda g_{\alpha'} \lambda h_{\alpha\alpha} \lambda x_\alpha(y_{\alpha'}(f_{\alpha'\alpha'} g_{\alpha'} h_{\alpha\alpha})(z_{\alpha'}(g_{\alpha'} h_{\alpha\alpha}) x_\alpha)).$
$\langle A_{\alpha'}, B_{\alpha'} \rangle \rightarrow \omega_{\alpha''\alpha'\alpha'} A_{\alpha'} B_{\alpha'}.$
$P_{\alpha'\alpha'''} \rightarrow \lambda n_{\alpha'''}(n_{\alpha'''}(\lambda p_{\alpha''} \langle S_{\alpha'\alpha'}(p_{\alpha''}(K_{\alpha'\alpha'\alpha'} I_{\alpha'}) 0_{\alpha'}),$
$p_{\alpha''}(K_{\alpha'\alpha'\alpha'} I_{\alpha'}) 0_{\alpha'} \rangle) \langle 0_{\alpha'}, 0_{\alpha'} \rangle (K_{\alpha'\alpha'\alpha'} 0_{\alpha'}) I_{\alpha'}.$
$T_{\alpha''\alpha'} \rightarrow \lambda x_{\alpha'}[(\imath x_{\alpha''})[(N_{o\alpha''} x_{\alpha''})[x_{\alpha''} S_{\alpha'\alpha'} 0_{\alpha'} = x_{\alpha'}]]].$
$P_{\alpha'\alpha'} \rightarrow \lambda x_{\alpha'}(P_{\alpha'\alpha'''}(T_{\alpha'''\alpha''}(T_{\alpha''\alpha'} x_{\alpha'}))).$

As a further abbreviation, we omit square brackets [ ] introduced by the above abbreviations, when possible without ambiguity. When, in omitting square brackets, the initial bracket is replaced by a bold dot **.**, it is to be understood that the scope of the omitted pair of brackets is from the dot forward the maximum distance which is consistent with the whole expression's being well-formed or interpretable as an abbreviation of a well-formed formula. When omitted brackets are not thus replaced by a dot, the convention in restoring omitted brackets is association to the left, except as modified by the understanding that the abbreviated formulas are well-formed and by the following relation of precedence among the different kinds of brackets. The brackets in $[\sim \boldsymbol{A}_o]$ and $[\boldsymbol{A}_o \boldsymbol{B}_o]$ are of lowest rank, those in $[(\boldsymbol{x}_\alpha)\boldsymbol{A}_o]$ and $[(\exists \boldsymbol{x}_\alpha)\boldsymbol{A}_o]$ and $[(\iota \boldsymbol{x}_\alpha)\boldsymbol{A}_o]$ and $[\boldsymbol{A}_\alpha = \boldsymbol{B}_\alpha]$ and $[\boldsymbol{A}_\alpha \neq \boldsymbol{B}_\alpha]$ are of next higher rank, those in $[\boldsymbol{A}_o \vee \boldsymbol{B}_o]$ are of next higher rank, and those in $[\boldsymbol{A}_o \supset \boldsymbol{B}_o]$ and $[\boldsymbol{A}_o \equiv \boldsymbol{B}_o]$ are of highest rank; in restoring omitted brackets (not represented by a dot), those of lower rank are to be put in before those of higher rank, so that the smaller scope is allotted to those of lower rank. For example,

$$\sim p \supset q \supset . \, pq \vee rs \supset \sim . \, q \vee s \supset \sim p \sim r$$

is an abbreviation for

$$[[[\sim p]\supset q]\supset[[[pq]\vee[rs]]\supset[\sim[[q\vee s]\supset[[\sim p][\sim r]]]]]],$$

which is in turn an abbreviation for

$$\begin{aligned}&((A_{ooo}(N_{oo}((A_{ooo}(N_{oo}(N_{oo}p_o)))q_o)))((A_{ooo}(N_{oo}((A_{ooo}(N_{oo}((A_{ooo}(N_{oo}p_o))(N_{oo}q_o))))\\ &\qquad(N_{oo}((A_{ooo}(N_{oo}r_o))(N_{oo}s_o))))))\\ &\qquad(N_{oo}((A_{ooo}(N_{oo}((A_{ooo}q_o)s_o)))(N_{oo}((A_{ooo}(N_{oo}(N_{oo}p_o)))(N_{oo}(N_{oo}r_o)))))))).\end{aligned}$$

In the intended interpretation of the formal system λ will have the rôle of an abstraction operator, N_{oo} will denote negation, A_{ooo} will denote disjunction, $\Pi_{o(o\alpha)}$ will denote the universal quantifier (as a propositional function of propositional functions), $\iota_{\alpha(o\alpha)}$ will denote a selection operator (as a function of propositional functions), and juxtaposition, between parentheses, will denote application of a function to its argument. Such a logical construction of the natural numbers in each type α' is intended that $0_{\alpha'}$ will denote the natural number 0, $1_{\alpha'}$ will denote 1, $2_{\alpha'}$ will denote 2, etc. Then $S_{\alpha'\alpha'}$ will denote the successor function of natural numbers; or, more exactly, it will denote a function which has the entire type α' as the range of its argument and which operates as a successor function in the case that the argument is a natural number. Moreover, $N_{o\alpha'}$ will denote the propositional function "to be a natural number (of type α')." If $\boldsymbol{N}_{\alpha''}$ denotes a natural number of type α'', then $\boldsymbol{N}_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}$ denotes the same (more exactly, the corresponding) natural number in the type α'. Hence if $\boldsymbol{N}_{\alpha'}$ denotes a natural number of type α', the same natural number in the type α'' will be denoted by $T_{\alpha''\alpha'}\boldsymbol{N}_{\alpha'}$. The formula $P_{\alpha'\alpha'''}$ is adapted from Kleene's formula P employed in the calculus of λ-conversion[7] and has the property that if $\boldsymbol{N}_{\alpha'''}$ denotes a natural number of type α''' then $P_{\alpha'\alpha'''}\boldsymbol{N}_{\alpha'''}$ denotes the predecessor of that natural number in the type α'. The true predecessor function, which gives the predecessor in the same type, is denoted by $P_{\alpha'\alpha'}$; it follows from the independence of the axiom of infinity (§4) that this predecessor function cannot be defined without using descriptions (i.e., the selection operator $\iota_{\beta(o\beta)}$).

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[7] S. C. Kleene, *A theory of positive integers in formal logic*, ***American journal of mathematics***, vol. 57 (1935), pp. 153–173, 219–244.

3. Rules of inference. The rules of inference (or rules of procedure) are the six following:

I. *To replace any part* $\boldsymbol{M}_\alpha$ *of a formula by the result of substituting* $\boldsymbol{y}_\beta$ *for* $\boldsymbol{x}_\beta$ *throughout* $\boldsymbol{M}_\alpha$, *provided that* $\boldsymbol{x}_\beta$ *is not a free variable of* $\boldsymbol{M}_\alpha$ *and* $\boldsymbol{y}_\beta$ *does not occur in* $\boldsymbol{M}_\alpha$. (I.e., to infer from a given formula the formula obtained by this replacement.)

II. *To replace any part* $((\lambda \boldsymbol{x}_\beta \boldsymbol{M}_\alpha)\boldsymbol{N}_\beta)$ *of a formula by the result of substituting* $\boldsymbol{N}_\beta$ *for* $\boldsymbol{x}_\beta$ *throughout* $\boldsymbol{M}_\alpha$, *provided that the bound variables of* $\boldsymbol{M}_\alpha$ *are distinct both from* $\boldsymbol{x}_\beta$ *and from the free variables of* $\boldsymbol{N}_\beta$.

III. *Where* $\boldsymbol{A}_\alpha$ *is the result of substituting* $\boldsymbol{N}_\beta$ *for* $\boldsymbol{x}_\beta$ *throughout* $\boldsymbol{M}_\alpha$, *to replace any part* $\boldsymbol{A}_\alpha$ *of a formula by* $((\lambda \boldsymbol{x}_\beta \boldsymbol{M}_\alpha)\boldsymbol{N}_\beta)$, *provided that the bound variables of* $\boldsymbol{M}_\alpha$ *are distinct both from* $\boldsymbol{x}_\beta$ *and from the free variables of* $\boldsymbol{N}_\beta$.

IV. *From* $\boldsymbol{F}_{o\alpha}\boldsymbol{x}_\alpha$ *to infer* $\boldsymbol{F}_{o\alpha}\boldsymbol{A}_\alpha$, *provided that* $\boldsymbol{x}_\alpha$ *is not a free variable of* $\boldsymbol{F}_{o\alpha}$.

V. *From* $\boldsymbol{A}_o \supset \boldsymbol{B}_o$ *and* $\boldsymbol{A}_o$, *to infer* $\boldsymbol{B}_o$.

VI. *From* $\boldsymbol{F}_{o\alpha}\boldsymbol{x}_\alpha$ *to infer* $\Pi_{o(o\alpha)}\boldsymbol{F}_{o\alpha}$, *provided that* $\boldsymbol{x}_\alpha$ *is not a free variable of* $\boldsymbol{F}_{o\alpha}$.

The word *part* of a formula is to be understood here as meaning *consecutive well-formed part* other than a variable immediately following an occurrence of λ. Moreover, as already explained, bold capital letters represent *well-formed* formulas and bold small letters represent variables, the subscript in each case showing the type. When (as in the rules I, II, III) we speak of replacing a part $\boldsymbol{M}_\alpha$ of a formula by something else, it is to be understood that, if there are several occurrences of $\boldsymbol{M}_\alpha$ as a part of the formula, any *one* of them may be so replaced. When we speak of the result of substituting $\boldsymbol{N}_\beta$ for $\boldsymbol{x}_\beta$ throughout $\boldsymbol{M}_\alpha$, the case is not excluded that $\boldsymbol{x}_\beta$ fails to occur in $\boldsymbol{M}_\alpha$, the result of the subtitution in that case being $\boldsymbol{M}_\alpha$.

The rules I–III are called rules of λ-*conversion*, and any chain of applications of these rules is called a λ-conversion, or briefly, a conversion. Rule IV is the rule of *substitution*, Rule V is the rule of *modus ponens*, and Rule VI is the rule of *generalization*. In an application of Rule IV, we say that the variable $\boldsymbol{x}_\alpha$ is *substituted for*; and in an application of Rule VI, we say that the variable $\boldsymbol{x}_\alpha$ is *generalized upon*.

The two following rules of inference are derived rules, in the sense that the indicated inference can be accomplished in each case by a chain of applications of I–VI (the effect of IV′ can be obtained by means of λ-conversion and Rule IV, the effect of VI′ can be obtained by means of λ-conversion and Rule VI):

IV′. *From* $\boldsymbol{M}_o$ *to infer the result of substituting* $\boldsymbol{A}_\alpha$ *for the free occurrences of* $\boldsymbol{x}_\alpha$ *throughout* $\boldsymbol{M}_o$, *provided that the bound variables of* $\boldsymbol{M}_o$ *other than* $\boldsymbol{x}_\alpha$ *are distinct from the free variables of* $\boldsymbol{A}_\alpha$.

VI′. *From* $\boldsymbol{M}_o$ *to infer* $(\boldsymbol{x}_\alpha)\boldsymbol{M}_o$.

4. Formal axioms. The formal axioms are the formulas in the following infinite list:

1. $p \vee p \supset p$.
2. $p \supset p \vee q$.

3. $p \vee q \supset q \vee p$.
4. $p \supset q \supset . r \vee p \supset r \vee q$.
5[α]. $\Pi_{o(o\alpha)} f_{o\alpha} \supset f_{o\alpha} x_\alpha$.
6[α]. $(x_\alpha)[p \vee f_{o\alpha} x_\alpha] \supset p \vee \Pi_{o(o\alpha)} f_{o\alpha}$.
7. $(\exists x_\iota)(\exists y_\iota) . x_\iota \neq y_\iota$.
8. $N_{o\iota'} x_{\iota'} \supset . N_{o\iota'} y_{\iota'} \supset . S_{\iota'\iota'} x_{\iota'} = S_{\iota'\iota'} y_{\iota'} \supset x_{\iota'} = y_{\iota'}$.
9[α]. $f_{o\alpha} x_\alpha \supset . (y_\alpha)[f_{o\alpha} y_\alpha \supset x_\alpha = y_\alpha] \supset f_{o\alpha}(\iota_{\alpha(o\alpha)} f_{o\alpha})$.
10[αβ]. $(x_\beta)[f_{\alpha\beta} x_\beta = g_{\alpha\beta} x_\beta] \supset f_{\alpha\beta} = g_{\alpha\beta}$.
11[α]. $f_{o\alpha} x_\alpha \supset f_{o\alpha}(\iota_{\alpha(o\alpha)} f_{o\alpha})$.

The *theorems* of the system are the formulas obtainable from the formal axioms by a succession of applications of the rules of inference. A *proof* of a theorem of the system is a finite sequence of formulas, the last of which is the theorem, and each of which is either a formal axiom or obtainable from preceding formulas in the sequence by an application of a rule of inference.

We must, of course, distinguish between *formal theorems*, or theorems of the system, and *syntactical theorems*, or theorems about the system, this and related distinctions being a necessary part of the process of using a known language (English) to set up another (more exact) language. (We deliberately use the word "theorem" ambiguously, sometimes for a proposition and sometimes for a sentence or formula meaning the proposition in some language.)

Axioms 1–4 suffice for the propositional calculus and Axioms 1–6$^\alpha$ for the logical functional calculus.

In order to obtain elementary number theory it is necessary to add (to 1–6$^\alpha$) Axioms 7, 8, and 9$^\alpha$. Of these, 9$^\alpha$ are *axioms of descriptions*, and 7 and 8 taken together have the effect of an *axiom of infinity*. The independence of Axiom 7 may be established by considering an interpretation of the primitive symbols according to which there is exactly one individual, and that of Axiom 8 by considering an interpretation according to which there are a finite number, more than one, of individuals.

In order to obtain classical real number theory (analysis) it is necessary[8] to add also Axioms 10$^{\alpha\beta}$ and 11$^\alpha$. Of these, 10$^{\alpha\beta}$ are *axioms of extensionality* for functions, and 11$^\alpha$ are *axioms of choice*.

Axioms 10$^{\alpha\beta}$, although weaker in some directions than axioms of extensionality which are sometimes employed, are nevertheless adequate. For classes may be introduced in such a way that the class associated with the propositional function denoted by $F_{o\alpha}$ is denoted by $\lambda x_\alpha(\imath y_{\iota'}) . (F_{o\alpha} x_\alpha)[y_{\iota'} = 0_{\iota'}] \vee (\sim F_{o\alpha} x_\alpha)[y_{\iota'} = 1_{\iota'}]$. We remark, however, on the possibility of introducing the additional axiom of extensionality, $p \equiv q \supset p = q$, which has the effect of imposing so broad a criterion of identity between propositions that there are in consequence only two propositions, and which, in conjunction with 10$^{\alpha\beta}$, makes possible the identification of classes with propositional functions.

Axioms 9$^\alpha$ obviously fail to be independent of 1–4 and 11$^\alpha$. We have never-

[8] Devices of contextual definition, such as Russell's methods of introducing classes and descriptions (loc. cit.), are here avoided, and assertions concerning the necessity of axioms and the like are to be understood in the sense of this avoidance.

theless included the axioms 9^α because of the desirability of considering the consequences of Axioms 1–9^α without $10^{\alpha\beta}$, 11^α.

If 1–9^α are the only formal axioms, each of the axioms 9^α is then independent, but if 10^α is added there is a sense in which those other than 9^o and 9^ι, although independent, are superfluous. For, of the symbols $\iota_{\alpha(o\alpha)}$, we may introduce only $\iota_{o(oo)}$ and $\iota_{\iota(o\iota)}$ as primitive symbols and then introduce the remainder by definition (i.e., by conventions of abbreviation) in such a way that the formulas 9^α, read in accordance with these definitions (conventions of abbreviation), become theorems provable from the formal axioms 1–8, 9^o, 9^ι, $10^{\alpha\beta}$. The required definitions are summarized in the following schema, which states the definition of $\iota_{\alpha\beta(o(\alpha\beta))}$ in terms of $\iota_{\alpha(o\alpha)}$:

$$\iota_{\alpha\beta(o(\alpha\beta))} \rightarrow \lambda h_{o(\alpha\beta)} \lambda x_\beta (\imath y_\alpha)(\exists f_{\alpha\beta}) \cdot h_{o(\alpha\beta)} f_{\alpha\beta} \cdot y_\alpha = f_{\alpha\beta} x_\beta.$$

5. The deduction theorem. Derivation of the formal theorems of the propositional calculus from Axioms 1–4 by means of Rules IV′ and V is well known and need not be repeated here.[9] In what follows we shall employ theorems of the propositional calculus as needed, assuming the proof as known.

It is also clear that, by means of Rules I and IV′, alphabetical changes of the variables (free and bound) may be made in any formal axiom, provided that the types of the variables are not altered, that variables originally the same remain the same, and that variables originally different remain different. Formal theorems obtained in this way (including the formal axioms themselves) will be called *variants* of the axioms and will be employed as needed without explicit statement of the proof.

By a *proof* of a formula $\boldsymbol{B}_o$ *on the assumption of* the formulas $\boldsymbol{A}_o^1, \boldsymbol{A}_o^2, \cdots, \boldsymbol{A}_o^n$, we shall mean a finite sequence of formulas, the last of which is $\boldsymbol{B}_o$, and each of which is either one of the formulas $\boldsymbol{A}_o^1, \boldsymbol{A}_o^2, \cdots, \boldsymbol{A}_o^n$, or a variant of a formal axiom, or obtainable from preceding formulas in the sequence by an application of a rule of inference subject to the condition that no variable shall be substituted for or generalized upon which appears as a free variable in any of the formulas $\boldsymbol{A}_o^1, \boldsymbol{A}_o^2, \cdots, \boldsymbol{A}_o^n$. In order to express that there is a proof of $\boldsymbol{B}_o$ on the assumption of $\boldsymbol{A}_o^1, \boldsymbol{A}_o^2, \cdots, \boldsymbol{A}_o^n$, we shall employ the (syntactical) notation:

$$\boldsymbol{A}_o^1, \boldsymbol{A}_o^2, \cdots, \boldsymbol{A}_o^n \vdash \boldsymbol{B}_o.$$

In the use of this notation, it is not excluded that n should be 0 and the set of formulas $\boldsymbol{A}_o^i$ vacuous; i.e., the notation $\vdash \boldsymbol{B}_o$ will be used to mean that $\boldsymbol{B}_o$ is a (formal) theorem. (This use of the sign $\vdash$ must be distinguished from the entirely different use of the assertion sign by Russell and earlier by Frege.)

The following syntactical theorem is known as the *deduction theorem*:

VII. *If* $\boldsymbol{A}_o^1, \boldsymbol{A}_o^2, \cdots, \boldsymbol{A}_o^n \vdash \boldsymbol{B}_o$, *then* $\boldsymbol{A}_o^1, \boldsymbol{A}_o^2, \cdots, \boldsymbol{A}_o^{n-1} \vdash \boldsymbol{A}_o^n \supset \boldsymbol{B}_o$ $(n = 1, 2, 3, \cdots)$.

In order to prove this, we suppose that the finite sequence of formulas $\boldsymbol{B}_o^1$, $\boldsymbol{B}_o^2, \cdots, \boldsymbol{B}_o^m$ is a proof of $\boldsymbol{B}_o$ on the assumption of $\boldsymbol{A}_o^1, \boldsymbol{A}_o^2, \cdots, \boldsymbol{A}_o^n$, the formula

[9] Cf. Hilbert and Ackermann, loc. cit.; P. Bernays, *Axiomatische Untersuchung des Aussagen-Kalkuls der "Principia Mathematica," Mathematische Zeitschrift*, vol. 25 (1926), pp. 305–320.

B_o^m being the same as B_o, and we show in succession, for each value of i from 1 to m, that

$$A_o^1, A_o^2, \cdots, A_o^{n-1} \vdash A_o^n \supset B_o^i.$$

This is done by cases, according as B_o^i is A_o^n, is one of $A_o^1, A_o^2, \cdots, A_o^{n-1}$, is a variant of an axiom, or is obtained from a preceding formula or pair of formulas by one of the rules I–VI. If B_o^i is A_o^n, we may obtain $A_o^n \supset B_o^i$ from $p \supset p$ by IV′. If B_o^i is one of $A_o^1, A_o^2, \cdots, A_o^{n-1}$ or is a variant of an axiom, we may obtain $A_o^n \supset B_o^i$ from $p \supset . q \supset p$ by a succession of applications of IV′ and V. If B_o^i is obtained from B_o^a $(a < i)$ by one of the rules I, II, III, we may obtain $A_o^n \supset B_o^i$ from $A_o^n \supset B_o^a$ by the same rule. If B_o^i is obtained from B_o^a $(a < i)$ by Rule IV, we may obtain $A_o^n \supset B_o^i$ from $A_o^n \supset B_o^a$ by IV′. If B_o^i is obtained from B_o^a and B_o^b $(a < i, b < i)$ by Rule V, we may obtain $A_o^n \supset B_o^i$ from $A_o^n \supset B_o^a$ and $A_o^n \supset B_o^b$ and $p \supset [q \supset r] \supset . p \supset q \supset . p \supset r$ by a succession of applications of IV′ and V. If B_o^i is obtained from B_o^a $(a < i)$ by Rule VI, we may obtain $A_o^n \supset B_o^i$ from $A_o^n \supset B_o^a$ and 6^α by a succession of applications of IV′, V, and VI′.

Proof of the following theorems,[10] which are consequences of the formal axioms 1–6^α, is left to the reader (it will be found convenient in most cases to abbreviate the proof by employing the deduction theorem in the rôle of a derived rule):

12^α. $(x_\alpha) f_{o\alpha} x_\alpha \supset f_{o\alpha} y_\alpha$.

13^α. $f_{o\alpha} y_\alpha \supset (\exists x_\alpha) f_{o\alpha} x_\alpha$.

14^α. $(x_\alpha)[p \supset f_{o\alpha} x_\alpha] \supset . p \supset (x_\alpha) f_{o\alpha} x_\alpha$.

15^α. $(x_\alpha)[f_{o\alpha} x_\alpha \supset p] \supset . (\exists x_\alpha) f_{o\alpha} x_\alpha \supset p$.

16^α. $x_\alpha = x_\alpha$.

17^α. $x_\alpha = y_\alpha \supset . f_{o\alpha} x_\alpha \supset f_{o\alpha} y_\alpha$.

$18^{\beta\alpha}$. $x_\alpha = y_\alpha \supset f_{\beta\alpha} x_\alpha = f_{\beta\alpha} y_\alpha$.

19^α. $x_\alpha = y_\alpha \supset y_\alpha = x_\alpha$.

20^α. $x_\alpha = y_\alpha \supset . y_\alpha = z_\alpha \supset x_\alpha = z_\alpha$.

The following theorems are consequences of the formal axioms 1–4 and $10^{\alpha\beta}$ (no use will be made of them below because we shall be concerned entirely with consequences of 1–9^α):

$21^{\alpha\beta}$. $f_{\alpha\beta} = \lambda x_\beta (f_{\alpha\beta} x_\beta)$.

6. Peano's postulates for arithmetic. Three of the five Peano postulates for arithmetic[11] are represented by the following formal theorems:

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[10] The same device of typical ambiguity which was employed in stating the rules of inference and formal axioms now serves us, not only to condense the statement of an infinite number of theorems (differing only in the type subscripts of the proper symbols which appear) into a single schema of theorems, but also to condense the proof of the infinite number of theorems into a single schema of proof. Of course, in the explicit formal development of the system, a stage would never be reached at which all of the theorems 12^o, 12^ι, $12^{\iota\iota}$, $\cdots$ (for example) had been proved, but by the device of a schema of proof with typical ambiguity we obtain metamathematical assurance that any required *one* of the theorems in the infinite list can be proved. Cf. the Prefatory Statement to the second volume of ***Principia mathematica***.

[11] G. Peano, *Sul concetto di numero*, ***Rivista di matematica***, vol. 1 (1891), pp. 87–102, 256–267.

22^α. $N_{o\alpha'}0_{\alpha'}$.

23^α. $N_{o\alpha'}x_{\alpha'} \supset N_{o\alpha'}(S_{\alpha'\alpha'}x_{\alpha'})$.

24^α. $f_{o\alpha'}0_{\alpha'} \supset . (x_{\alpha'})[N_{o\alpha'}x_{\alpha'} \supset . f_{o\alpha'}x_{\alpha'} \supset f_{o\alpha'}(S_{\alpha'\alpha'}x_{\alpha'})] \supset . N_{o\alpha'}x_{\alpha'} \supset f_{o\alpha'}x_{\alpha'}$.

These theorems are consequences of 1–6^α; proofs are left to the reader.

From 24^α and the deduction theorem we obtain the following syntactical theorem which we shall call the *induction theorem*:

VIII. *If* $\mathbf{x}_{\alpha'}$ *is not a free variable of* $\mathbf{A}_o^1, \mathbf{A}_o^2, \cdots, \mathbf{A}_o^n, \mathbf{F}_{o\alpha'}$, *if* $\mathbf{A}_o^1, \mathbf{A}_o^2, \cdots, \mathbf{A}_o^n \vdash \mathbf{F}_{o\alpha'}0_{\alpha'}$, *and if* $\mathbf{A}_o^1, \mathbf{A}_o^2, \cdots, \mathbf{A}_o^n, N_{o\alpha'}\mathbf{x}_{\alpha'}, \mathbf{F}_{o\alpha'}\mathbf{x}_{\alpha'} \vdash \mathbf{F}_{o\alpha'}(S_{\alpha'\alpha'}\mathbf{x}_{\alpha'})$, *then* $\mathbf{A}_o^1, \mathbf{A}_o^2, \cdots, \mathbf{A}_o^n \vdash N_{o\alpha'}\mathbf{x}_{\alpha'} \supset \mathbf{F}_{o\alpha'}\mathbf{x}_{\alpha'}$.

A proof which is or can be abbreviated by employing the induction theorem in the rôle of a derived rule will be called a proof by (*mathematical*, or *complete*) *induction* on the variable $\mathbf{x}_{\alpha'}$.

Another of the Peano postulates is represented by the following formal theorems:

25^α. $N_{o\alpha'}x_{\alpha'} \supset S_{\alpha'\alpha'}x_{\alpha'} \neq 0_{\alpha'}$.

These theorems are consequences of 1–6^α and 7, as we shall show (for certain types α they are consequences of 1–6^α only).

The remaining Peano postulate would correspond to the following:

26^α. $N_{o\alpha'}x_{\alpha'} \supset . N_{o\alpha'}y_{\alpha'} \supset . S_{\alpha'\alpha'}x_{\alpha'} = S_{\alpha'\alpha'}y_{\alpha'} \supset x_{\alpha'} = y_{\alpha'}$.

These formulas are demonstrably not theorems (consistency assumed) in the case of type symbols α consisting entirely of o's with no ι's. We shall show that the formulas 26^ι, $26^{\iota'}$, $26^{\iota''}$, $\cdots$ are theorems—in fact they are consequences of 1–6^α and 8, the formula 26^ι being the same as 8.

A proof of the theorem,

27^o. $(\exists x_o)(\exists y_o) . x_o \neq y_o$,

may be made as follows. In 17^o substitute $p \vee \sim p$ for x_o, and $\sim . p \vee \sim p$ for y_o and $\lambda r \sim . p \vee \sim p \supset \sim r$ for f_{oo}, by successive applications of IV′, and then apply Rule II twice, so obtaining

$$[p \vee \sim p] = [\sim . p \vee \sim p] \supset . [\sim . p \vee \sim p \supset \sim . p \vee \sim p] \supset \sim . p \vee \sim p \supset \sim\sim . p \vee \sim p.$$

Hence using the theorems of the propositional calculus,

$$\sim . p \vee \sim p \supset \sim . p \vee \sim p,$$

$$q \supset [r \supset s] \supset . r \supset . q \supset s,$$

and the rules IV′ and V, obtain

$$[p \vee \sim p] = [\sim . p \vee \sim p] \supset \sim . p \vee \sim p \supset \sim\sim . p \vee \sim p.$$

Hence, using the theorems of the propositional calculus,

$$p \vee \sim p \supset \sim\sim . p \vee \sim p,$$

$$q \supset . r \supset \sim q \supset \sim r,$$

and IV′ and V (method of *reductio ad absurdum*), obtain

$$[p \vee \sim p] \neq [\sim . p \vee \sim p].$$

Hence by two successive uses of 13^o, with I, II, III, IV′, V, obtain 27^o.

In regard to proof of the theorems,

27^α. $(\exists x_\alpha)(\exists y_\alpha) . x_\alpha \neq y_\alpha$,

since we have a proof of 27^o, and 27^ι is Axiom 7, it is sufficient to show how to obtain a proof of $27^{\alpha\beta}$ if a proof of 27^α is given.

By conversion $z_\alpha \neq t_\alpha \vdash K_{\alpha\beta\alpha} z_\alpha x_\beta \neq K_{\alpha\beta\alpha} t_\alpha x_\beta$.

Hence by 17^α (using II, IV′, V), $z_\alpha \neq t_\alpha, K_{\alpha\beta\alpha} z_\alpha = K_{\alpha\beta\alpha} t_\alpha \vdash K_{\alpha\beta\alpha} t_\alpha x_\beta \neq K_{\alpha\beta\alpha} t_\alpha x_\beta$.

Hence by the deduction theorem, $z_\alpha \neq t_\alpha \vdash K_{\alpha\beta\alpha} z_\alpha = K_{\alpha\beta\alpha} t_\alpha \supset K_{\alpha\beta\alpha} t_\alpha x_\beta \neq K_{\alpha\beta\alpha} t_\alpha x_\beta$.

By 16^α (using IV′), $K_{\alpha\beta\alpha} t_\alpha x_\beta = K_{\alpha\beta\alpha} t_\alpha x_\beta$.

Hence by *reductio ad absurdum*, as above, $z_\alpha \neq t_\alpha \vdash K_{\alpha\beta\alpha} z_\alpha \neq K_{\alpha\beta\alpha} t_\alpha$.

Hence by two successive uses of $13^{\alpha\beta}$ (with I, II, III, IV′, V), $z_\alpha \neq t_\alpha \vdash (\exists x_{\alpha\beta})(\exists y_{\alpha\beta}) . x_{\alpha\beta} \neq y_{\alpha\beta}$.

Hence by the deduction theorem, $\vdash z_\alpha \neq t_\alpha \supset (\exists x_{\alpha\beta})(\exists y_{\alpha\beta}) . x_{\alpha\beta} \neq y_{\alpha\beta}$.

Hence using VI′, $\vdash (t_\alpha) . z_\alpha \neq t_\alpha \supset (\exists x_{\alpha\beta})(\exists y_{\alpha\beta}) . x_{\alpha\beta} \neq y_{\alpha\beta}$.

Hence by 15^α (using I, II, III, IV′, V), $\vdash (\exists t_\alpha)[z_\alpha \neq t_\alpha] \supset (\exists x_{\alpha\beta})(\exists y_{\alpha\beta}) . x_{\alpha\beta} \neq y_{\alpha\beta}$.

Hence using VI′, $\vdash (z_\alpha) . (\exists t_\alpha)[z_\alpha \neq t_\alpha] \supset (\exists x_{\alpha\beta})(\exists y_{\alpha\beta}) . x_{\alpha\beta} \neq y_{\alpha\beta}$.

Hence by 15^α (using I, II, III, IV′, V), $\vdash (\exists z_\alpha)(\exists t_\alpha)[z_\alpha \neq t_\alpha] \supset (\exists x_{\alpha\beta})(\exists y_{\alpha\beta}) . x_{\alpha\beta} \neq y_{\alpha\beta}$.

Hence if $\vdash 27^\alpha$ then, using I and V, $\vdash 27^{\alpha\beta}$.

Thus for every type α we have a proof of 27^α. Using this, we proceed to the proof of

28^α. $S_{\alpha'\alpha'} x_{\alpha'} \neq 0_{\alpha'}$.

By conversion, $z_\alpha \neq t_\alpha \vdash S_{\alpha'\alpha'} x_{\alpha'} (K_{\alpha\alpha\alpha} z_\alpha) t_\alpha \neq 0_{\alpha'} (K_{\alpha\alpha\alpha} z_\alpha) t_\alpha$. Hence by the method illustrated in the preceding proof, using in order 17^α, the deduction theorem, 16^α, and *reductio ad absurdum*, $z_\alpha \neq t_\alpha \vdash S_{\alpha'\alpha'} x_{\alpha'} \neq 0_{\alpha'}$. Eliminating the assumption $z_\alpha \neq t_\alpha$ by the method of the preceding proof, using in order the deduction theorem, VI′, 15^α, VI′, 15^α, 27^α, we have $\vdash 28^\alpha$.

Having 28^α, we prove 25^α by using $p \supset . q \supset p$.

We need also the theorems:

29^α. $N_{o\alpha''} n_{\alpha''} \supset N_{o\alpha'}(n_{\alpha''} S_{\alpha'\alpha'} 0_{\alpha'})$.

The (schema of) proof of these theorems is a simple example of proof by induction.

From 22^α by conversion, $\vdash N_{o\alpha'}(0_{\alpha''} S_{\alpha'\alpha'} 0_{\alpha'})$.

By 23^α, $N_{o\alpha'}(n_{\alpha''} S_{\alpha'\alpha'} 0_{\alpha'}) \vdash N_{o\alpha'}(S_{\alpha'\alpha'}(n_{\alpha''} S_{\alpha'\alpha'} 0_{\alpha'}))$.

Hence by conversion, $N_{o\alpha'}(n_{\alpha''} S_{\alpha'\alpha'} 0_{\alpha'}) \vdash N_{o\alpha'}(S_{\alpha''\alpha''} n_{\alpha''} S_{\alpha'\alpha'} 0_{\alpha'})$.

Hence by the induction theorem, taking $F_{o\alpha''}$ to be $\lambda x_{\alpha''}(N_{o\alpha'}(x_{\alpha''} S_{\alpha'\alpha'} 0_{\alpha'}))$ and $x_{\alpha''}$ to be $n_{\alpha''}$, and employing conversion as required, we have $\vdash 29^\alpha$.

Returning now to 26^α, we consider in connection with it:

$$30^\alpha.\quad N_{o\alpha''}m_{\alpha''} \supset . N_{o\alpha''}n_{\alpha''} \supset . m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset m_{\alpha''} = n_{\alpha''}.$$

As in the case of 26^α, not all the formulas 30^α are theorems. We shall show that 26^α and 30^α are theorems if α is one of the types $\iota, \iota', \iota'', \cdots$. Since 26^ι is Axiom 8, we may do this by showing that (1) if $\vdash 26^\alpha$ then $\vdash 30^\alpha$, and (2) if $\vdash 26^\alpha$ and $\vdash 30^\alpha$ then $\vdash 26^{\alpha'}$.[12]

By $18^{\alpha'\alpha''}$, $S_{\alpha''\alpha''}x_{\alpha''} = S_{\alpha''\alpha''}y_{\alpha''} \vdash S_{\alpha''\alpha''}x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha''\alpha''}y_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}$.

Hence by conversion, we have $S_{\alpha''\alpha''}x_{\alpha''} = S_{\alpha''\alpha''}y_{\alpha''} \vdash S_{\alpha'\alpha'}(x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}) = S_{\alpha'\alpha'}(y_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'})$.

Hence if $\vdash 26^\alpha$, we have by 29^α, $N_{o\alpha''}x_{\alpha''}$, $N_{o\alpha''}y_{\alpha''}$, $S_{\alpha''\alpha''}x_{\alpha''} = S_{\alpha''\alpha''}y_{\alpha''} \vdash x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = y_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}$.

Hence if $\vdash 26^\alpha$ and $\vdash 30^\alpha$, we have $N_{o\alpha''}x_{\alpha''}$, $N_{o\alpha''}y_{\alpha''}$, $S_{\alpha''\alpha''}x_{\alpha''} = S_{\alpha''\alpha''}y_{\alpha''} \vdash x_{\alpha''} = y_{\alpha''}$.

Hence by three applications of the deduction theorem, if $\vdash 26^\alpha$ and $\vdash 30^\alpha$ then $\vdash 26^{\alpha'}$. This is (2) above.

Now by conversion, $0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = 0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \vdash 0_{\alpha'} = 0_{\alpha'}$. Hence by the deduction theorem, $\vdash 0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = 0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset 0_{\alpha'} = 0_{\alpha'}$.

By conversion, $0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha''\alpha''}n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \vdash 0_{\alpha'} = S_{\alpha'\alpha'}(n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'})$. Hence by $19^{\alpha'}$, $0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha''\alpha''}n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \vdash S_{\alpha'\alpha'}(n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}) = 0_{\alpha'}$. By 28^α, $\vdash S_{\alpha'\alpha'}(n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}) \neq 0_{\alpha'}$. Hence, using $p\sim p\supset q$, we have $0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha''\alpha''}n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha} \vdash 0_{\alpha''} = S_{\alpha''\alpha''}n_{\alpha''}$. Hence by the deduction theorem, $\vdash 0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha''\alpha''}n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset 0_{\alpha''} = S_{\alpha''\alpha''}n_{\alpha}$.

Hence by the induction theorem, followed by VI′, $\vdash (n_{\alpha''}) . N_{o\alpha''}n_{\alpha''} \supset . 0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset 0_{\alpha''} = n_{\alpha''}$.

By conversion, $S_{\alpha''\alpha''}m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = 0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \vdash S_{\alpha'\alpha'}(m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}) = 0_{\alpha'}$. By 28^α, $\vdash S_{\alpha'\alpha'}(m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}) \neq 0_{\alpha'}$. Hence, using $p\sim p\supset q$, we have $S_{\alpha''\alpha''}m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = 0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \vdash S_{\alpha''\alpha''}m_{\alpha''} = 0_{\alpha''}$. Hence by the deduction theorem, $\vdash S_{\alpha''\alpha''}m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = 0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset S_{\alpha''\alpha''}m_{\alpha''} = 0_{\alpha''}$.

By conversion, $S_{\alpha''\alpha''}m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha''\alpha''}n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \vdash S_{\alpha'\alpha'}(m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}) = S_{\alpha'\alpha'}(n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'})$. Hence if $\vdash 26^\alpha$, we have by 29^α, $N_{o\alpha''}m_{\alpha''}$, $N_{o\alpha''}n_{\alpha''}$, $S_{\alpha''\alpha''}m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha''\alpha''}n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \vdash m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}$. Hence if $\vdash 26^\alpha$, we have (using $12^{\alpha''}$), $N_{o\alpha''}m_{\alpha''}$, $(n_{\alpha''}) . N_{o\alpha''}n_{\alpha''} \supset . m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset m_{\alpha''} = n_{\alpha''}$, $N_{o\alpha''}n_{\alpha''}$, $S_{\alpha''\alpha''}m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha''\alpha''}n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \vdash m_{\alpha''} = n_{\alpha''}$. Hence, using $18^{\alpha''\alpha''}$ to obtain $S_{\alpha''\alpha''}m_{\alpha''} = S_{\alpha''\alpha''}n_{\alpha''}$ and then applying the deduction theorem, we have (if $\vdash 26^\alpha$), $N_{o\alpha''}m_{\alpha''}$, $(n_{\alpha''}) . N_{o\alpha''}n_{\alpha''} \supset . m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset m_{\alpha''} = n_{\alpha''}$, $N_{o\alpha''}n_{\alpha''} \vdash S_{\alpha''\alpha''}m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha''\alpha''}n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset S_{\alpha''\alpha''}m_{\alpha''} = S_{\alpha''\alpha''}n_{\alpha''}$.

Hence by the induction theorem, followed by VI′, we have (if $\vdash 26^\alpha$), $N_{o\alpha''}m_{\alpha''}$,

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[12] The question suggests itself whether 30^ι could be used in place of Axiom 8 as the second part of the axiom of infinity. The writer has a proof (depending on the properties of $P_{\iota'\iota''}$) that 30^ι and $30^{\iota'}$ are together sufficient, in the presence of $1\text{–}6^\alpha$, to replace Axiom 8. A proof has also been carried out by A. M. Turing that, in the presence of 1–7 and 9^α, 30^ι is sufficient alone to replace Axiom 8. Whether 8 is independent of 1–7 and 30^ι remains an open problem (familiar methods of eliminating descriptions do not apply here).

$(n_{\alpha''}) . N_{o\alpha''}n_{\alpha''} \supset . m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset m_{\alpha''} = n_{\alpha''} \vdash (n_{\alpha''}) . N_{o\alpha''}n_{\alpha''} \supset . S_{\alpha''\alpha''}m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset S_{\alpha''\alpha''}m_{\alpha''} = n_{\alpha''}$.

Hence again, applying the induction theorem to preceding results, we have (if $\vdash 26^{\alpha}$), $\vdash N_{o\alpha''}m_{\alpha''} \supset . (n_{\alpha''}) . N_{o\alpha''}n_{\alpha''} \supset . m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset m_{\alpha''} = n_{\alpha''}$.

Hence using V and $12^{\alpha''}$, we have (if $\vdash 26^{\alpha}$), $N_{o\alpha''}m_{\alpha''}, N_{o\alpha''}n_{\alpha''} \vdash m_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = n_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} \supset m_{\alpha''} = n_{\alpha''}$.

Hence by two applications of the deduction theorem, if $\vdash 26^{\alpha}$ then $\vdash 30^{\alpha}$. This is (1) above.

7. Properties of $T_{\alpha''\alpha'}$. We proceed now to proofs of the following theorems:

31^{α}. $N_{o\alpha'}x_{\alpha'} \supset N_{o\alpha''}(T_{\alpha''\alpha'}x_{\alpha'})$.

32^{α}. $N_{o\alpha'}x_{\alpha'} \supset T_{\alpha''\alpha'}x_{\alpha'}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'}$.

The proofs require $9^{\alpha''}$ and are possible only for types α for which there is a proof of 30^{α}.

We begin by proving as a lemma:

33^{α}. $N_{o\alpha'}x_{\alpha'} \supset (\exists x_{\alpha''}) . N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'}$.

Proof of this requires only the axioms 1–6^{α} and is possible for an arbitrary type α.

By 16^{α}, using IV′ and conversion, we have $\vdash 0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = 0_{\alpha'}$. Hence using $22^{\alpha'}$ and $p \supset . q \supset pq$ and $13^{\alpha''}$, we have $\vdash (\exists x_{\alpha''}) . N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = 0_{\alpha'}$.

By $18^{\alpha'\alpha'}$, $x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'} \vdash S_{\alpha'\alpha'}(x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}) = S_{\alpha'\alpha'}x_{\alpha'}$. Hence by conversion, $x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'} \vdash S_{\alpha''\alpha''}x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha'\alpha'}x_{\alpha'}$. Also, by $23^{\alpha'}$, $N_{o\alpha''}x_{\alpha''} \vdash N_{o\alpha''}(S_{\alpha''\alpha''}x_{\alpha''})$. Hence using $pq \supset p$ and $pq \supset q$ and $p \supset . q \supset pq$, we have $N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'} \vdash N_{o\alpha''}(S_{\alpha''\alpha''}x_{\alpha''}) . S_{\alpha''\alpha''}x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha'\alpha'}x_{\alpha'}$. Hence employing in order $13^{\alpha''}$, the deduction theorem, and VI′, we have $\vdash (x_{\alpha''}) . [N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'}] \supset (\exists x_{\alpha''}) . N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha'\alpha'}x_{\alpha'}$. Hence by $15^{\alpha''}$, $\vdash (\exists x_{\alpha''})[N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'}] \supset (\exists x_{\alpha''}) . N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha'\alpha'}x_{\alpha'}$.

Hence by the induction theorem, $\vdash 33^{\alpha}$.

Now proceeding with the proof of 31^{α} and 32^{α} (for types α for which $\vdash 30^{\alpha}$), we may—with the aid of 30^{α}—show that $N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'}, N_{o\alpha''}y_{\alpha''} . y_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'} \vdash x_{\alpha''} = y_{\alpha''}$.

Hence by the deduction theorem and VI′, $N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'} \vdash (y_{\alpha''}) . [N_{o\alpha''}y_{\alpha''} . y_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'}] \supset x_{\alpha''} = y_{\alpha''}$.

Hence by $9^{\alpha''}$ (refer to the definition of $T_{\alpha''\alpha'}$, §2), $N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'} \vdash N_{o\alpha''}(T_{\alpha''\alpha'}x_{\alpha'}) . T_{\alpha''\alpha'}x_{\alpha'}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'}$.

Hence employing in order the deduction theorem, VI′, and $15^{\alpha''}$, we have $\vdash (\exists x_{\alpha''})[N_{o\alpha''}x_{\alpha''} . x_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'}] \supset . N_{o\alpha''}(T_{\alpha''\alpha'}x_{\alpha'}) . T_{\alpha''\alpha'}x_{\alpha'}S_{\alpha'\alpha'}0_{\alpha'} = x_{\alpha'}$.

Hence using 33^{α} and $p \supset q \supset . [q \supset rs] \supset . p \supset r$ and $p \supset q \supset . [q \supset rs] \supset . p \supset s$, we have $\vdash 31^{\alpha}$ and $\vdash 32^{\alpha}$.

A further property of $T_{\alpha''\alpha'}$ is contained in the following theorem (if α is a type for which there is a proof of 30^{α}):

34^{α}. $N_{o\alpha'}x_{\alpha'} \supset T_{\alpha''\alpha'}(S_{\alpha'\alpha'}x_{\alpha'}) = S_{\alpha''\alpha''}(T_{\alpha''\alpha'}x_{\alpha'})$.

Proof of this depends on using $23^{\alpha'}$ to prove $N_{o\alpha''}(S_{\alpha''\alpha''}(T_{\alpha''\alpha'}x_{\alpha'}))$ on the assumption of $N_{o\alpha'}x_{\alpha'}$, and using $16^{\alpha'}$ and conversion to prove $S_{\alpha''\alpha''}(T_{\alpha''\alpha'}x_{\alpha'})S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha'\alpha'}(T_{\alpha''\alpha'}x_{\alpha'}S_{\alpha'\alpha'}0_{\alpha'})$ and hence $S_{\alpha''\alpha''}(T_{\alpha''\alpha'}x_{\alpha'})S_{\alpha'\alpha'}0_{\alpha'} = S_{\alpha'\alpha'}x_{\alpha'}$ on the assumption of $N_{o\alpha'}x_{\alpha'}$—then using 30^{α} (with 23^{α}, 32^{α}, 31^{α}).

A similar use of 30^{α} leads to a proof of the following (where α is a type for which there is a proof of 30^{α}):

35^{α}. $T_{\alpha''\alpha'}0_{\alpha'} = 0_{\alpha''}$.

8. Definition by primitive recursion. The formalization of definition by primitive recursion requires that, given formulas $\boldsymbol{A}_{\alpha'}$ and $\boldsymbol{B}_{\alpha'\alpha'\alpha'}$, we find a formula $\boldsymbol{F}_{\alpha'\alpha'}$ such that the following are theorems (where $\boldsymbol{x}_{\alpha'}$ is not a free variable of $\boldsymbol{A}_{\alpha'}$, $\boldsymbol{B}_{\alpha'\alpha'\alpha'}$, or $\boldsymbol{F}_{\alpha'\alpha'}$):

$$\boldsymbol{F}_{\alpha'\alpha'}0_{\alpha'} = \boldsymbol{A}_{\alpha'}.$$

$$N_{o\alpha'}\boldsymbol{x}_{\alpha'} \supset \boldsymbol{F}_{\alpha'\alpha'}(S_{\alpha'\alpha'}\boldsymbol{x}_{\alpha'}) = \boldsymbol{B}_{\alpha'\alpha'\alpha'}\boldsymbol{x}_{\alpha'}(\boldsymbol{F}_{\alpha'\alpha'}\boldsymbol{x}_{\alpha'}).$$

This may be done by taking $\boldsymbol{F}_{\alpha'\alpha'}$ to be the following formula (where $\boldsymbol{x}_{\alpha'}$, $\boldsymbol{y}_{\alpha''}$ are not free variables of $\boldsymbol{A}_{\alpha'}$ or $\boldsymbol{B}_{\alpha'\alpha'\alpha'}$):[13]

$$\lambda\boldsymbol{x}_{\alpha'} \,.\, T_{\alpha'''\alpha''}(T_{\alpha''\alpha'}\boldsymbol{x}_{\alpha'})(\lambda\boldsymbol{y}_{\alpha''}\langle S_{\alpha'\alpha'}(\boldsymbol{y}_{\alpha''}(K_{\alpha'\alpha'\alpha'}I_{\alpha'})0_{\alpha'}),$$
$$\boldsymbol{B}_{\alpha'\alpha'\alpha'}(\boldsymbol{y}_{\alpha''}(K_{\alpha'\alpha'\alpha'}I_{\alpha'})0_{\alpha'})(\boldsymbol{y}_{\alpha''}(K_{\alpha'\alpha'\alpha'}0_{\alpha'})I_{\alpha'})\rangle)\langle 0_{\alpha'}, \boldsymbol{A}_{\alpha'}\rangle(K_{\alpha'\alpha'\alpha'}0_{\alpha'})I_{\alpha'}.$$

The definition of $P_{\alpha'\alpha'}$ already given is a particular case and may be used as an illustration. The following theorems may be proved in order:

36^{α}. $N_{o\alpha'}n_{\alpha'} \supset \lambda f_{\alpha\alpha}\lambda x_{\alpha}(n_{\alpha'}f_{\alpha\alpha}x_{\alpha}) = n_{\alpha'}$. (By induction, using $16^{\alpha'}$, $18^{\alpha'\alpha'}$, and conversion.)

37^{α}. $N_{o\alpha'}m_{\alpha'} \supset .\, N_{o\alpha'}n_{\alpha'} \supset \langle m_{\alpha'}, n_{\alpha'}\rangle(K_{\alpha'\alpha'\alpha'}I_{\alpha'})0_{\alpha'} = m_{\alpha'}$. (By induction on $n_{\alpha'}$, using 36^{α}.)

38^{α}. $N_{o\alpha'}m_{\alpha'} \supset .\, N_{o\alpha'}n_{\alpha'} \supset \langle m_{\alpha'}, n_{\alpha'}\rangle(K_{\alpha'\alpha'\alpha'}0_{\alpha'})I_{\alpha'} = n_{\alpha'}$. (By induction on $m_{\alpha'}$, using 36^{α}.)

39^{α}. $N_{o\alpha'''}n_{\alpha'''} \supset n_{\alpha'''}(\lambda p_{\alpha''}\langle S_{\alpha'\alpha'}(p_{\alpha''}(K_{\alpha'\alpha'\alpha'}I_{\alpha'})0_{\alpha'}),\ p_{\alpha''}(K_{\alpha'\alpha'\alpha'}I_{\alpha'})0_{\alpha'}\rangle)\langle 0_{\alpha'},\ 0_{\alpha'}\rangle(K_{\alpha'\alpha'\alpha'}I_{\alpha'})0_{\alpha'} = n_{\alpha'''}S_{\alpha''\alpha''}0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}$. (By induction, using 29^{α}, 37^{α}.)

40^{α}. $N_{o\alpha'''}n_{\alpha'''} \supset P_{\alpha'\alpha'''}(S_{\alpha'''\alpha'''}n_{\alpha'''}) = n_{\alpha'''}S_{\alpha''\alpha''}0_{\alpha''}S_{\alpha'\alpha'}0_{\alpha'}$. (By 39^{α}, 38^{α}, using 29^{α}.)

41^{α}. $P_{\alpha'\alpha'''}0_{\alpha'''} = 0_{\alpha'}$. (By $16^{\alpha'}$ and conversion.)

42^{α}. $P_{\alpha'\alpha'}0_{\alpha'} = 0_{\alpha'}$. (By 41^{α}, $35^{\alpha'}$, 35^{α}.)

43^{α}. $N_{o\alpha'}n_{\alpha'} \supset P_{\alpha'\alpha'}(S_{\alpha'\alpha'}n_{\alpha'}) = n_{\alpha'}$. (By 40^{α}, $31^{\alpha'}$, 31^{α}, $34^{\alpha'}$, 34^{α}, $32^{\alpha'}$, 32^{α}.)

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[13] This schema employs descriptions, through the appearance in it of $T_{\alpha'''\alpha''}$ and $T_{\alpha''\alpha'}$. In certain cases a formula $\boldsymbol{F}_{\alpha'\alpha'}$ may be obtained which does not involve descriptions. In particular, for addition and multiplication of non-negative integers we may use the definitions due to J. B. Rosser:

$s_{\alpha'\alpha'\alpha'} \rightarrow \lambda m_{\alpha'}\lambda n_{\alpha'}\lambda f_{\alpha\alpha}\lambda x_{\alpha}(m_{\alpha'}f_{\alpha\alpha}(n_{\alpha'}f_{\alpha\alpha}x_{\alpha}))$.

$B_{\alpha'\alpha'\alpha'} \rightarrow \lambda m_{\alpha'}\lambda n_{\alpha'}\lambda f_{\alpha\alpha}(m_{\alpha'}(n_{\alpha'}f_{\alpha\alpha}))$.

$[\boldsymbol{A}_{\alpha'} + \boldsymbol{B}_{\alpha'}] \rightarrow s_{\alpha'\alpha'\alpha'}\boldsymbol{A}_{\alpha'}\boldsymbol{B}_{\alpha'}$.

$[\boldsymbol{A}_{\alpha'} \times \boldsymbol{B}_{\alpha'}] \rightarrow B_{\alpha'\alpha'\alpha'}\boldsymbol{A}_{\alpha'}\boldsymbol{B}_{\alpha'}$.